Multiple-soliton solutions of Einstein's equations

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Using the Belinsky–Zakharov generating technique and a flat metric as a seed, two- and four-soliton solutions of the Einstein vacuum equations for the cases of stationary axisymmetric, cylindrically symmetric, or plane symmetric gravitational fields are considered. Three- and five-parameter classes of exact solutions are obtained, some of which are new.

I. INTRODUCTION

Among the techniques developed in recent years for the generation of new solutions of the Einstein vacuum (and stiff matter or Einstein–Maxwell) equations from simpler known ones the inverse scattering method (ISM) of Belinsky and Zakharov[1] (BZ) has turned out to be one of the most fertile. Because it applies in all cases where the space-time manifold admits a pair of commuting non-null Killing vectors, the BZ technique has already been used for the construction of a large number of exact solutions representing stationary axisymmetric, cylindrically symmetric, and plane symmetric gravitational fields.[2–7]

In the present paper we consider the "2- and 2×2 soliton" solutions of the Einstein vacuum equations, which can be constructed using the BZ technique and a simple (flat) Kasner metric as the known or "seed" solution. This type of diagonal seed was first used by BZ in illustrative examples of the application of their method in the original papers cited above. Where we differ from BZ and other authors who have used the Kasner or other diagonal seed metrics is in treating the three cases mentioned below in a unified manner. Thus following the lead of Letelier,[3] we first develop a simplified version of the BZ formulas for the product metric coefficients for the general "N-soliton" solution in terms of determinants of $N\times N$ matrices. Subsequently, we specialize our results to the two-soliton case and use the method of Tomitatsu[8] to make them applicable to the four-soliton double poles or the two$\times$two-soliton case as well. The application of these results to the simple Kasner metric mentioned above allows us to accomplish the following objectives.

First, we rederive several important solutions of the Einstein vacuum equations recently discovered by other methods. These include the Chandrasekhar–Xanthopoulos[9] colliding plane-waves solution; the cosmic string plus gravitational waves solution of Xanthopoulos[10]; and the five-parameter family of stationary axisymmetric metrics discovered by Kinnersley and Chitre,[11] which generalized the $\delta=2$ solution of Tomimatsu and Sato.[12] There are two points gained by this rederivation. On one hand, the interrelations between the above solutions are clearly brought out and a frame of their classification is established. On the other hand, the advantage of the BZ technique of giving all the components of the product metric tensor by algebraic means is made explicit. Thus the BZ technique allows us to construct the Kinnersley–Chitre[11] metric completely, while the method by which this solution was arrived at originally allowed for the construction of the corresponding Ernst potential only.

The second objective accomplished in the applications part of this paper consists of generating several classes of new solutions, including (i) one-parameter generalizations of the Chandrasekhar–Xanthopoulos[9] and Xanthopoulos[10] metrics, and (ii) six families of pentaparametric solutions (the Kinnersley–Chitre metric being one of them)—two in each of the stationary axisymmetric, cylindrically symmetric, and plane symmetric groups of space-times.

The plan of our paper is as follows. In Sec. II, we present an outline of the BZ solution-generating method. In Sec. III, we give a set of determinantal formulas for the product metric coefficients which hold in the general N-soliton case when the seed metric is diagonal. On the basis of the above formulas, the two-soliton product metric coefficients are constructed in Sec. IV in terms of the two "pole trajectory" functions and a pair of arbitrary real or complex parameters. Section IV also covers the method by which the formulas given in Sec. III are made applicable in the "double-poles" case. Last, in Sec. V, we present the new solutions of the Einstein equations that can be obtained as two- and two$\times$two-soliton products of the application of the results of Sec. IV on the Kasner seed case.

II. THE BZ SOLITON TECHNIQUE

The metric of the space-times under consideration can be written in the form

$$ds^2=f(x^3,x^4)[-\epsilon(dx^4)^2+(dx^3)^2] + g_{ab}(x^3,x^4)dx^a\,dx^b\,, \tag{2.1}$$

where a, b run from 1 to 2 and $\epsilon=\pm 1$.

Introducing the coordinates ζ, η, and the function α via

$$\zeta\equiv x^3+\sqrt{\epsilon}x^4\,,\quad \eta=x^3-\sqrt{\epsilon}x^4\,, \tag{2.2}$$

and

$$\alpha^2=\det(g)\equiv\det(g_{ab})\,, \tag{2.3}$$

respectively, one can write the Einstein vacuum equations for the metric (2.1) in the form

$$(\alpha g_{,\zeta}g^{-1})_{,\eta}+(\alpha g_{,\eta}g^{-1})_{,\zeta}=0\,, \tag{2.4}$$

$$(\ln f)_{,\zeta}=(\ln\alpha)_{,\zeta\zeta}/(\ln\alpha)_{,\zeta}+\mathrm{Tr}(\alpha g_{,\zeta}g^{-1})^2/4\alpha\,\alpha_{,\zeta}\,, \tag{2.5a}$$

$$(\ln f)_{,\eta}=(\ln\alpha)_{,\eta\eta}/(\ln\alpha)_{,\eta}+\mathrm{Tr}(\alpha g_{,\eta}g^{-1})^2/4\alpha\,\alpha_{,\eta}\,, \tag{2.5b}$$

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where $(\ )_x \equiv \partial_x(\ ) \equiv \partial(\ )/\partial x$. In particular, the trace of Eq. (2.3) reads as

$$\alpha_{,\zeta\eta} = 0 , \qquad (2.6)$$

whose general solution can be given in the form

$$\alpha(\zeta,\eta) = a(\zeta) + b(\eta) , \qquad (2.7)$$

where a, b are arbitrary functions of the indicated arguments.

In the BZ approach, Eq. (2.4), which is the integrability condition of Eqs. (2.5), is replaced by the coupled "Schrödinger equations"

$$D_\zeta \psi \equiv \left(\partial_\zeta + \frac{2\alpha_{,\zeta}\lambda}{\alpha - \lambda}\partial_\lambda\right)\psi = \frac{\alpha g_{,\zeta} g^{-1}}{(\alpha - \lambda)}\psi , \qquad (2.8a)$$

$$D_\eta \psi \equiv \left(\partial_\eta + \frac{2\alpha_{,\eta}\lambda}{\alpha - \lambda}\partial_\lambda\right)\psi = \frac{\alpha g_{,\eta} g^{-1}}{(\alpha - \lambda)}\psi \qquad (2.8b)$$

for the 2×2 matrix "wavefunction" ψ, which depends on the complex "spectral parameter" λ in such a way that

$$\lim_{\lambda\to 0} \psi(\zeta,\eta,\lambda) = g(\zeta,\eta) . \qquad (2.9)$$

Suppose now that $\psi^{(0)}$ is a known solution of Eqs. (2.8) which corresponds to $g^{(0)}$ in the sense of Eq. (2.9). Then, as BZ have shown, the ansatz

$$\psi = X\psi^{(0)} , \qquad (2.10)$$

together with the assumption that the "scattering matrix" $X(\zeta,\eta,\lambda)$ has only simple poles in the complex λ plane, leads to a new solution of Eqs. (2.8) which, thanks to Eq. (2.9), determines a new solution of Eq. (2.4).

The poles μ_k, $k = 1,\ldots,N$ of the scattering matrix X are found to be coordiante dependent. Specifically, the pole trajectories are given by

$$\mu_k = (w_k - \beta) + [(w_k - \beta)^2 - \alpha^2]^{1/2} , \qquad (2.11)$$

where w_k are arbitrary constants and

$$\beta = a(\zeta) - b(\eta) \qquad (2.12)$$

is the harmonic conjugate of the function α defined by Eq. (2.7).

It turns out that the functions $\mu_k(\zeta,\eta)$, along with the matrix $\psi^{(0)}$, determine the new metric (f,g) produced from the seed $(f^{(0)}, g^{(0)})$ completely and only via algebraic manipulations. This is made explicit in the final BZ formulas, which read as

$$f = c f^{(0)} \alpha^{-N^2/2} \left|\prod_{k=1}^{N} \mu_k\right|^{N+1} \times \left[\prod_{\substack{k,l=1\\k>l}}^{N} (\mu_k - \mu_l)^2\right]^{-1} \det(\Gamma) , \qquad (2.13a)$$

$$g_{ab} = \left[\prod_{k=1}^{N} |\mu_k/\alpha|\right] \times \left[g_{ab}^{(0)} - \sum_{k,l=1}^{N} \frac{\Gamma_{kl}^{-1} N_a^{(k)} N_b^{(l)}}{\mu_k \mu_l}\right] , \qquad (2.13b)$$

where

$$\Gamma_{kl} \equiv (\mu_k\mu_l - \alpha^2)^{-1} n_a^{(k)} g_{ab}^{(0)} n_b^{(l)} , \qquad (2.14a)$$

$$N_a^{(k)} \equiv g_{ab}^{(0)} n_b^{(k)} , \qquad (2.14b)$$

$$n_a^{(k)} \equiv m_b^{(k)} M_{ba}^{(k)} , \quad m_b^{(k)} \text{ arbitrary constants,} \qquad (2.14c)$$

$$M^{(k)} \equiv [\psi^{(0)}(\eta,\zeta;\mu_k)]^{-1} , \qquad (2.14d)$$

and c is an arbitrary real constant.

In Eqs. (2.14) the summation convention holds for the indices a, b, while in Eq. (2.13a) the term in square brackets should be replaced by unity when $N = 1$.

III. SOLUTIONS DERIVABLE FROM A DIAGONAL SEED

It is clear from Eqs. (2.14) that the key item in the construction of the new solution (f,g) of Einstein's vacuum equations starting from the "seed metric" $(f^{(0)},g^{(0)})$ is the set of N matrices $\{\psi^{(0)}(\zeta,\eta;\mu_k)\}$: These are obtained by integrating the system of equations (2.8) along the pole trajectories. However, when $g^{(0)}$ is not diagonal, the system (2.8) is generally very difficult to integrate. Therefore, one usually adopts the simplifying assumption that $g^{(0)}$ is a diagonal matrix, in which case $\psi^{(0)}$ can be assumed to be diagonal as well. Integrating the trace of Eqs. (2.8) along the pole trajectories μ_k we obtain

$$\det(\psi^{(0)}) = 2w_k\mu_k , \qquad (3.1)$$

which allows us to write the matrix $\psi^{(0)}$ in the form

$$\psi^{(0)}(\zeta,\eta;\mu_k) = \mathrm{diag}(\psi_k, 2w_k\mu_k\psi_k^{-1}) , \qquad (3.2)$$

where the function ψ_k satisfies

$$(\ln\psi_k)_{,\zeta} = [\alpha/(\alpha - \mu_k)](\ln g_{11}^{(0)})_{,\zeta} , \qquad (3.3a)$$

$$(\ln\psi_k)_{,\eta} = [\alpha/(\alpha + \mu_k)](\ln g_{11}^{(0)})_{,\eta} . \qquad (3.3b)$$

The diagonality assumption for $g^{(0)}$ simplifies not only the procedure that leads to the matrix $\psi^{(0)}$, but the algebraic system of Eqs. (2.13) and (2.14) as well. Thus substituting Eq. (3.2) into Eqs. (2.13), one finds that

$$g_{11} = \left|\prod_k \sigma_k\right| \left\{1 - \sum_{k,l}\left(\frac{S_k S_l}{\sigma_k\sigma_l}\right)\Delta_{kl}^{-1}\right\} g_{11}^{(0)} , \qquad (3.4a)$$

$$g_{22} = \left|\prod_k \sigma_k\right| \left\{1 - \sum_{k,l}\left(\frac{1}{\sigma_k\sigma_l}\right)\Delta_{kl}^{-1}\right\} g_{22}^{(0)} , \qquad (3.4b)$$

$$g_{12} = -\left(\frac{\alpha}{\sqrt{\epsilon}}\right)\left|\prod_k \sigma_k\right| \sum_{k,l}\left(\frac{S_k}{\sigma_k\sigma_l}\right)\Delta_{kl}^{-1} , \qquad (3.4c)$$

where

$$\sigma_k \equiv (\mu_k/\alpha) , \qquad (3.5a)$$

$$S_k \equiv q_k g_{11}^{(0)} \psi_k^{-2} \sigma_k , \quad q_k \text{ arbitrary constants,} \qquad (3.5b)$$

$$\Delta_{k,l} \equiv (S_k S_{l+1})/(\sigma_k\sigma_l - 1) , \qquad (3.5c)$$

and all the sums and products run from 1–N. Finally, the identities

$$\det(\gamma_k\delta_l + \Delta_{kl}) = \left(1 + \sum_{k,l}\gamma_k\delta_l\Delta_{kl}^{-1}\right)\det(\Delta) , \qquad (3.6a)$$

$$\det(\gamma_k\delta_l\Delta_{kl}) = \left(\prod_k \gamma_k\right)^2 \det(\Delta) \qquad (3.6b)$$

allow us to write Eq. (3.4) in the determinantal form (to be compared with the third paper listed in Ref. 3)

$$g_{11} = \left|\prod_{k=1}^{N} \sigma_k\right| \frac{L_{(-1)}}{L_{(0)}} g_{11}^{0} , \qquad (3.7a)$$

$$g_{22}=\epsilon^N\left|\prod_{k=1}^{N}\sigma_k^{-1}\right|\frac{L_{(+1)}}{L_{(0)}}g_{22}^{(0)}, \tag{3.7b}$$

$$g_{12}=\frac{\alpha}{\sqrt{\epsilon}}\left|\prod_{k=1}^{N}\sigma_k\right|\frac{L}{L_{(0)}}, \tag{3.7c}$$

where

$$L_{(\delta)}\equiv\det\Delta_{(\delta)}\ ,\quad \Delta_{(\delta)kl}\equiv\left[\frac{(\sigma_k\sigma_l)^{\delta}S_kS_l+1}{\sigma_k\sigma_l-1}\right], \tag{3.8a}$$

with $\delta=0,\ \pm1$ and

$$L\equiv\det(\Delta_{(0)kl})-\det[\Delta_{(0)kl}+(S_k/\sigma_k\sigma_l)]\ . \tag{3.8b}$$

Similarly, the metric coefficient f can be written in the form

$$f=c\frac{\alpha^{-N^2/2}\left|\Pi_{k=1}^{N}\sigma_k\right|^N}{\left[\Pi_{k,l=1,\ k>l}^{N}(\sigma_k-\sigma_l)^2\right][\Pi_{k=1}^{N}S_k]}L_{(0)}f^{(0)}. \tag{3.9}$$

Before concluding this section, let us note that the product metric, given by Eqs. (3.7)–(3.9), depends on $2N$ parameters—the arbitrary constants w_k and q_k. The w_k determine the pole trajectories μ_k via Eq. (2.9) and are incorporated in the function σ_k according to Eq. (3.5a). The q_k appear as multiplicative constants in the functions S_k defined by Eq. (3.5b). Thus one can make the functions S_k vanish by setting all the q_k's equal to zero. According to Eq. (3.8b) this choice leads to a diagonal product metric and therefore, it represents the easiest application of the BZ formulas. A diagonal product metric is also obtained if we let all the (q_k^{-1})'s go to zero. Such diagonal N-soliton metrics were given and studied by Carr and Verdaguer for the case where the Kasner cosmological model serves as seed.[5] In the general nondiagonal case, the functions S_k depend on the ψ_k's; the latter are obtained by integrating Eqs. (3.3). Assuming that the diagonal seed metric is written in the form

$$g^{(0)}=\mathrm{diag}(\alpha/\sqrt{\epsilon})(\epsilon e^{\phi},e^{-\phi})\ , \tag{3.10}$$

one can obtain the S_k functions more directly from the expression

$$\ln S_k=-\int\frac{(\ln\sigma_k)_{,\zeta}}{(\ln\alpha)_{,\zeta}}\phi_{,\zeta}\,d\zeta+\frac{(\ln\sigma_k)_{,\eta}}{(\ln\alpha)_{,\eta}}\phi_{,\eta}\,d\eta\ , \tag{3.11}$$

which results by combining Eqs. (3.3), (3.5a), and (3.5b). Still, specific applications of the BZ technique can be carried to completion only by assuming simple expressions for the function ϕ, which determines the g_{ab} part of the seed metric.

IV. TWO- AND TWO×TWO-SOLITON SOLUTIONS

A. A pair of simple poles

Let us now assume that the scattering matrix X has only two simple poles in the complex λ plane, located at w_1 and w_2, respectively. Then Eqs. (3.7)–(3.9) and simple algebra give the following expressions for the general two-soliton solution derivable from a diagonal seed:

$$g_{11}=[(\sigma_1\sigma_2-1)^2(\sigma_1S_2-\sigma_2S_1)^2+(\sigma_1-\sigma_2)^2\times(\sigma_1\sigma_2+S_1S_2)^2]\,Zg_{11}^{(0)}\ , \tag{4.1a}$$

$$g_{22}=[(\sigma_1\sigma_2-1)^2(\sigma_1S_1-\sigma_2S_2)^2+(\sigma_1-\sigma_2)^2\times(1+\sigma_1\sigma_2S_1S_2)^2]\,Zg_{22}^{(0)}\ , \tag{4.1b}$$

$$g_{12}=2(w_2-w_1)\sigma_1\sigma_2[(\sigma_1S_1-\sigma_2S_2)(\sigma_1\sigma_2+S_1S_2)+(\sigma_1S_2-\sigma_2S_1)(1+\sigma_1\sigma_2S_1S_2)]\,Z\ , \tag{4.1c}$$

$$f=\mathrm{const}[\sigma_1\sigma_2S_1S_2(\sigma_1^2-1)(\sigma_2^2-1)Z]^{-1}f^{(0)}\ , \tag{4.1d}$$

where

$$Z^{-1}\equiv|\sigma_1\sigma_2|[(\sigma_1-\sigma_2)^2(1+S_1S_2)^2+(\sigma_1\sigma_2-1)^2(S_1-S_2)^2]\ . \tag{4.2}$$

In deriving Eqs. (4.1c) and (4.1d) we made use of the identity

$$\alpha(\sigma_i-\sigma_j)(\sigma_i\sigma_j-1)=2(w_i-w_j)\sigma_i\sigma_j\ , \tag{4.3}$$

which follows from Eq. (2.11), and constant quantities were absorbed in the multiplicative constant c of Eq. (3.9).

At this point, let it be noted that according to Eqs. (2.11) and (3.5a),

$$\sigma_k^{(+)}\sigma_k^{(-)}=1\ \ (\text{no sum over } k)\ , \tag{4.4}$$

where $\sigma_k^{(\pm)}$ denote the σ_k functions corresponding to the $(\pm)$ choice of sign in Eq. (2.11). Similarly, Eq. (3.11) implies the relation

$$S_k^{(+)}S_k^{(1)}=D_k\ , \tag{4.5}$$

with D_k an arbitrary constant, between the $S_k^{(\pm)}$ functions that correspond to the $\sigma_k^{(\pm)}$'s, as in Eq. (3.5b). When relations (4.4) and (4.5) are taken into account, it is an easy matter to verify that the rhs of Eqs. (4.1) are invariant under the transformation

$$\{\sigma_j^{(+)},S_j^{(+)}\}\rightarrow\{\sigma_j^{(-)}=1/\sigma_j^{(+)},S_j^{(-)}=-1/S_j^{(+)}\}\ . \tag{4.6}$$

Therefore, any choice of sign in Eq. (2.11) implies no loss of generality.

On the other hand, the poles w_1, w_2 must both be real or complex conjugate. Therefore, there is no loss of generality if we take

$$w_1=-w_2=w,\quad \text{when } w_k\in\mathbb{R}, \tag{4.7a}$$

$$w_1=w_2=iw,\quad \text{when } w_k\in\mathbb{C}\ , \tag{4.7b}$$

since this choice implies no more than a translation in the $(\alpha/\sqrt{\epsilon},\beta)$ plane. Correspondingly, the pole trajectories can be chosen to be

$$\begin{aligned}\mu_1&=(w-\beta)+[(w-\beta)^2-\alpha^2]^{1/2}\ ,\\ \mu_2&=-(w+\beta)+[(w+\beta)^2-\alpha^2]^{1/2}\ ,\end{aligned} \tag{4.8}$$

when $w_k\in\mathbb{R}$ and

$$\mu_1=\mu_2=(iw-\beta)+[(w-\beta)^2-\alpha^2]^{1/2} \tag{4.9}$$

where $w_k\in\mathbb{C}$.

The expressions obtained thus far for the N-soliton solution are given in terms of the real-valued harmonic conjugate functions $(\alpha/\sqrt{\epsilon},\beta)$ which can be retained as the coordinate system replacing the original (x^3,x^4) or (ζ,η) system. However, in the two-soliton case, it turns out to be much more convenient to introduce the coordinates (x,y) defined by

$$\beta=wxy\ ,$$

$$\frac{\alpha}{\sqrt{\epsilon}}=\begin{cases}w[\epsilon(1-x^2)(1-y^2)]^{1/2}, & \text{when } w_k\in\mathbb{R}\ ,\\ w[\epsilon(1+x^2)(y^2-1)]^{1/2}, & \text{when } w_k\in\mathbb{C}\ .\end{cases} \tag{4.10}$$

Substituting Eqs. (4.8) and (4.10) into (3.10a), we find that

$$\sigma_1 = (1+x)(1-y)/[(1-x^2)(1-y^2)]^{1/2}, \quad \sigma_2 = (x-1)(1-y)/[(1-x^2)(1-y^2)]^{1/2} \tag{4.11}$$

when $w_k \in \mathbb{R}$. In order to obtain the corresponding σ_k's when $w_k \in \mathbb{C}$ one can simply use the mapping

$$(x, y;w) \to (-ix, y;iw):\text{(real poles case)} \to \text{(complex poles case)}, \tag{4.12}$$

which is implicit in Eq. (4.10).

B. A pair of double poles

The BZ formulas for the N-soliton solution are immediately applicable only when the N poles are distinct. Thus when two or more of the w_k's coincide one has to turn to the use of limiting procedures in order to find the appropriate version of the above formulas.

Consider, for example, the case where both w_1 and w_2 of Sec. IV A are double poles: By this we mean the case where the scattering matrix has four simple poles and we let $(w_3,w_4) \to (w_1,w_2)$. We then turn to Eqs. (3.7)–(3.9) in order to obtain the coefficients (f,g) of the new metric. Note, however, that if we let $(\sigma_3,\sigma_4) \to (\sigma_1,\sigma_2)$ and $(S_3,S_4) \to (S_1,S_2)$ as $(w_3,w_4) \to (w_1,w_2)$, then the four-soliton $[4\times4]$ matrix $\Delta_{(\delta)ij}$ will have pairs of equal rows and columns. As a result, the L functions will vanish, making Eqs. (3.7) and (3.9) inapplicable.

Therefore, let us consider the alternative[8] where $\sigma_1 = \sigma_1^{(+)}$, $\sigma_2 = \sigma_2^{(+)}$ and as $(w_3,w_4) \to (w_1,w_2)$:

$$(\sigma_3,\sigma_4) \to (\sigma_1^{(-)},\sigma_2^{(-)})$$

and

$$(S_3,S_4) \to (S_1^{(-)} = D_1/S_1^{(+)}, S_2^{(-)} = D_2/S_2^{(+)}), \tag{4.13}$$

where D_i's are arbitrary constants. The problem that now arises for the [1,3], [2,4] elements of the symmetric $[4\times4]$ matrices $\Delta_{(\delta)ij}$ is overcome by letting the D_i's go to -1. Therefore, let us consider

$$D_i = -1 + \xi_i(w_{i+2} - w_i), \tag{4.14}$$

where $i = 1,2$ and ξ_i are arbitrary constants. Then using Eqs. (3.15a), (4.8), (4.9), and (4.13) we find that the limit of $\Delta_{(\delta)ij}$ as $(w_3,w_4) \to (w_1,w_2)$ is the symmetric matrix $E_{(\delta)ij}$, where

$$E_{(\delta)kl} = \Delta_{(\delta)kl}, \tag{4.15a}$$

$$E_{(\delta)k+2,l+2} = -(\sigma_k\sigma_l)^{1-\delta}(S_kS_l)^{-1}\Delta_{(\delta)kl}, \tag{4.15b}$$

$$E_{(\delta)13} = -\left[\frac{\partial(\ln\sigma_1)}{\partial w_1}\right]^{-1}\left[\xi_1 + \frac{\partial(\ln S_1)}{\partial w_1}\right] - \delta, \tag{4.15c}$$

$$E_{(\delta)14} = \sigma_2^{1-\delta}S_2^{-1}(\sigma_2-\sigma_1)^{-1}(\sigma_1^\delta S_1 - \sigma_2^\delta S_2), \tag{4.15d}$$

$$E_{(\delta)23} = \sigma_1^{1-\delta}S_1^{-1}(\sigma_2-\sigma_1)^{-1}(\sigma_1^\delta S_1 - \sigma_2^\delta S_2), \tag{4.15e}$$

$$E_{(\delta)24} = -\left[\frac{\partial(\ln\sigma_2)}{\partial w_2}\right]^{-1}\left[\xi_2 + \frac{\partial(\ln S_2)}{\partial w_2}\right] - \delta, \tag{4.15f}$$

where k, l run from 1–2. Moreover, from Eqs. (2.9) and (3.5a) it easily follows that

$$\frac{\partial(\ln\sigma_k)}{\partial w_k} = \frac{2\sigma_k}{\alpha(\sigma_k^2 - 1)}. \tag{4.16}$$

Thus the equality of the pairs of rows or columns of $\Delta_{(\delta)ij}$ in the limit $(w_3,w_4) \to (w_1,w_2)$ has been broken and, once the functions S_k have been determined, Eqs. (4.15) and (4.16) provide the elements necessary for calculating the L functions of the four-soliton formulas.

V. APPLICATIONS

A. A pair of simple poles

As a first application of the results obtained in Sec. IV, let us consider the two-soliton solutions that can be derived from the metric

$$ds^2 = -\epsilon(dx^4)^2 + (dx^3)^2 + \epsilon(dx^1)^2 + (\alpha(x^3,x^4)\sqrt{\epsilon})^2(dx^2)^2, \tag{5.1}$$

where $(\alpha/\sqrt{\epsilon})$ is any real solution of Eq. (2.6).

The metric (5.1) results from taking

$$\phi = -\ln(\alpha/\sqrt{\epsilon}) \tag{5.2}$$

in Eq. (3.10), a choice that is in accord with the vacuum field equations (2.4). Substitution of Eq. (5.2) into (3.11) leads to the relation

$$S_k = Q_k\sigma_k, \quad Q_k \text{ arbitrary constants,} \tag{5.3}$$

which makes explicit the simplicity of expression (5.2) for ϕ from the standpoint of the soliton technique.

By choosing the value of ϵ and the specific form of α one specifies the "gauge." Thus we will distinguish the following cases.

(i) For the axisymmetric gauge,

$$\epsilon = -1, \quad (x^1,x^2,x^3,x^4) = (t,\varphi,z,\rho), \quad \alpha = i\rho, \quad \beta = z. \tag{5.4}$$

In this case Eq. (5.1) becomes

$$ds^2 = -dt^2 + d\rho^2 + \rho^2\, d\varphi^2 + dz^2, \tag{5.5}$$

which makes the range and meaning of the coordinates evident.

(ii) For the plane symmetric gauge,

$$\epsilon = 1, \quad (x^1,x^2,x^3,x^4) = (x',y',z',t'), \quad \alpha = t', \quad \beta = z'. \tag{5.6}$$

Now, the flat metric in Eq. (5.1) can be considered to represent a Kasner or Bianchi type I universe.

(iii) For the cylindrically symmetric gauge,

$$\epsilon = 1, \quad (x^1,x^2,x^3,x^4) = (z,\varphi,\rho,t), \quad \alpha = \rho, \quad \beta = t. \tag{5.7}$$

As in the axisymmetric gauge, in this case the seed metric is also the Minkowski metric in cylindrical coordinates. However, the product metric will be different. In the present case the two-soliton solution will preserve the cylindrical symmetry of the Minkowsi space-time, while in the axisymmetric gauge it is the stationary axially symmetric character of the original metric that will be preserved.

Returning to Eq. (5.3), let us note that the resulting metric will be real provided that Q_1 and Q_2 are chosen to be real or complex conjugate when the poles w_1 and w_2 are real or complex conjugate, respectively. In the real poles case, let us introduce the parameters p, q, and l via the relations

$$\frac{1}{q} = -\frac{Q_1+Q_2}{1+Q_1Q_2},\quad \frac{p}{q} = \frac{Q_1-Q_2}{1+Q_1Q_2},\quad \frac{l}{q} = \frac{Q_1Q_2-1}{Q_1Q_2+1}. \tag{5.8}$$

Then

$$q^2+p^2 = 1+l^2. \tag{5.9}$$

In order to cover the complex poles case, all we need do is let $p\to ip$ in Eq. (5.8); then (5.9) is replaced by

$$q^2-p^2 = 1+l^2. \tag{5.10}$$

1. Complex conjugate poles

Let us now substitute Eq. (5.3) into Eq. (4.1) and consider the case of complex w_k first. Taking into account the $p\to ip$ version of Eq. (5.8) we find that the product metric can be written in the form

$$ds^2 = f[-\epsilon(dx^4)^2 + (dx^3)^2] + \epsilon(Y/X)[dx^1 - \omega\, dx^2]^2 + (\alpha^2/\epsilon)(X/Y)(dx^2)^2, \tag{5.11}$$

where

$$f = \mathrm{const}[X/(x^2+y^2)], \tag{5.12a}$$

$$X = (l-qy)^2 + (1-px)^2, \tag{5.12b}$$

$$Y = q^2(y^2-1) + p^2(x^2+1), \tag{5.12c}$$

$$\left(\frac{\omega}{2w}\right) = -\frac{q(y^2-1)(p+x)+p(x^2+1)(q-ly)}{q^2(y^2-1)p^2(x^2+1)} = \frac{1}{p}\,\frac{q(y^2-1)(1-px+l^2-lq)+lp^2(y-1)(x^2+1)}{q^2(y^2-1)+p^2(x^2+1)} - \frac{(q-l)}{p}. \tag{5.12d}$$

Depending on the gauge, the line element given by Eqs. (5.11) and (5.12) represents the following three classes of space-times.

(i) The $a^2> m^2$ Kerr–NUT (Newman–Unti–Tamburino) metrices. This can be made explicit by first gauging away the constant $2p^{-1}(q-l)$ in the second version of Eq. (5.12d) by letting $x^1\to x^1 + 2p^{-1}(q-l)x^2$ and then choosing the axisymmetric gauge.

Inverting Eq. (4.10) we obtain

$$2wx = r_+ + r_-,\quad 2iwy = r_+ - r_-, \tag{5.13}$$

where

$$r_\pm = [(z+iw)^2+\rho^2]^{1/2}. \tag{5.14}$$

Equation (4.10) also gives the restriction $|y|<1$ for the range of the y coordinate. The asymptotic form of the metric shows that one must choose the ratio (w/p) such that

$$w = -mp, \tag{5.15}$$

where m is the mass parameter, while the arbitrary constant figuring in Eq. (5.12a) must be taken to be equal to p^{-2}. Substituting Eq. (5.15) into (5.10) one obtains

$$w = (a^2-m^2-b^2)^{1/2}, \tag{5.16}$$

where

$$a\equiv qm,\quad b\equiv lm. \tag{5.17}$$

Finally, by introducing the coordinates (r,θ) via the relations

$$wx = r-m,\quad y=\cos\theta \tag{5.18}$$

one obtains the Boyer–Linquist form of the Kerr–NUT metric, whereby a and b are seen to stand for the angular momentum and NUT parameter, respectively.

The derivation of the $a^2>m^2$ Kerr–NUT solution along the lines described above was obtained by BZ as one of the first applications of their ISM.[1]

(ii) Gravitational solitons propagating in a Kasner universe. When the plane symmetric gauge is chosen, Eq. (4.10) gives

$$2wx = r_+ + r_-,\quad 2iwy = r_+ - r_-,\quad r_\pm \equiv [(z\pm iw)^2 - t^2]^{1/2} \tag{5.19}$$

and the metric given by Eqs. (5.11) and (5.12) represents a pair of gravitational solitons propagating along opposite directions of the z axis. The solitons converge if the Kasner background is collapsing, i.e., for $t\in(-\infty,0)$, or diverge if the universe is expanding, i.e., for $t\in(0,\infty)$. This family of solutions was also first obtained by BZ.[1]

(iii) Cylindrical gravitational waves reflecting off the symmetry axis. In the cylindrically symmetric gauge Eq. (4.10) gives

$$2wx = r_+ + r_-,\quad 2iwy = r_+ - r_-,\quad r_\pm \equiv [(t+iw)^2-\rho^2]^{1/2}. \tag{5.20}$$

As in the axially symmetric case, the metric is easily regularized on the axis by gauging away the constant term in the second version of Eq. (5.12d). As shown by Economou and Tsoubelis,[7] the solution that is obtained in this fashion represents a solitary gravitational wave which, having started from $\rho\to\infty$ at $t\to-\infty$, reaches near the symmetry axis $\rho=0$ and reflects from it at $t=0$.

In the present case one can choose the arbitrary constant in the expression for the metric coeffficient f to be different from p^{-2}. Then the axis region is characterized by an angle deficit and the solution can be interpreted as a gravitational wave interacting with a cosmic string which occupies the axis of symmetry.

The $l=0$ subclass of cylindrically symmetric solutions given by Eqs. (5.11) and (5.12) was first obtained by Xanthopoulos[10] using a nonsolitonic technique. In fact, the whole class of solutions under consideration retains the Petrov type

D character of the Kerr–NUT metric and therefore, must be a member of the Kinnersley[13] family of solutions.

On the other hand, Letelier[3] has obtained a family of cylindrically symmetric solutions using the BZ ISM and a diagonal seed. Because the final expressions are very complicated and depend heavily on the gauge functions, it would have been very hard for us to check if the class of metrics presented above is contained in the Letelier family of solutions.

2. Real poles

As noted in Sec. IV A, the metric coefficients for the $w_k \in \mathbb{R}$ case are obtained from the ones corresponding to $w_k \in \mathbb{C}$ by the substitution $(x,w,p) \to (ix, -iw, -ip)$. Thus when $w_k \in \mathbb{R}$, Eq. (4.10) and its inverse read as

$$\alpha/\sqrt{\epsilon} = w[\epsilon(1-x^2)(1-y^2)]^{1/2}, \quad \beta = wxy \quad (5.21)$$

and

$$2wx = r_+ + r_-, \quad 2wy = r_+ - r_-, \quad r_\pm \equiv [(\beta \pm w)^2 - \alpha^2]^{1/2}, \quad (5.22)$$

respectively. Similarly, Eq. (5.12) becomes, in this case,

$$f = \text{const}[X/(x^2 - y^2)], \quad (5.23a)$$

$$X = (l - qy)^2 + (1 - px)^2, \quad (5.23b)$$

$$Y = p^2(x^2 - 1) + q^2(y^2 - 1), \quad (5.23c)$$

$$(\omega/2w) = -(pY)^{-1}[q(1 - px + l^2 - lq)(1 - y^2) + lp^2(1 - y)(1 - x^2)] - p^{-1}(q - l). \quad (5.23d)$$

The line element given by Eqs. (5.11) and (5.23) corresponds to the following classes of space-times.

(i) The Kerr–NUT $m^2 > a^2$ solutions. This class of solutions is obtained in the axisymmetric gauge ($\epsilon = -1$). Let

$$\alpha/i = \rho = w[(x^2 - 1)(1 - y^2)]^{1/2}, \quad \beta = z = wxy \quad (5.24)$$

and

$$w = (m^2 - a^2 + b^2)^{1/2}, \quad (5.25)$$

with a, b as in Eq. (5.17). Choosing the arbitrary constant that figures in Eq. (5.23a) to be equal to p^{-2} again, we have in Eqs. (5.11) and (5.23) the Kerr–NUT $m^2 > a^2$ metrics either in the Weyl normal coordinates (ρ,z) or the prolate spheroidal coordinates (x, y). In the latter case the coordinate patch consists of the strip $x \in (1,\infty)$, $y \in (-1,1)$. In terms of the Boyer–Lindquist coordinates (r,θ) defined by Eq. (5.18), this strip corresponds to the region $r > r_1 \equiv m + w$, i.e., to that part of space-time that lies outside the event horizon.

(ii) Colliding plane waves. In the plane symmetric gauge Eq. (4.10) becomes

$$\alpha = t = w[(1 - x^2)(1 - y^2)]^{1/2}, \quad \beta = z = wxy. \quad (5.26)$$

Thus the metric given by Eqs. (5.11) and (5.23) is real in those regions of the (x, y) plane where either $|x| < 1$ and $|y| < 1$ or $|x| > 1$ and $|y| > 1$. On the other hand, according to Eq. (5.23a) these regions are bisected by straight lines along which the metric coefficient f is singular. This implies that having chosen the metric in any one of the above regions, one must determine a well-defined process of continuing it beyond the boundaries. For example, let $w = -|w|$ in Eq. (5.26) and consider the interior of the triangle defined by the points (0,0), (1,1), and $(-1,1)$ of the (x, y) plane. The corresponding region in the (t,z) plane is bounded by the lines $t = 0$ and $t = -|w| \pm z$. As shown by Chandrasekhar and Xanthopoulos,[9] if one assumes that the space-time metric in this triangular region is the one defined by Eqs. (5.11) and (5.23), with $l = 0$, then for $t < 0$, one can extend it beyond the lines $t = -|w| \pm z$ using the Khan–Penrose[14] technique. The resulting solution represents gravitational plane waves which collide at $t = -|w|$ and Eqs. (5.11) and (5.23) give the metric in the region of the waves' interaction.

Exactly in the same way, one can extend the general $l \neq 0$ metric and verify that the resulting metric again represents collision of gravitational plane waves. We note here that this metric can be obtained from the $l = 0$ case by an application of the hyperbolic version of Ehlers transformation. In fact, Ernst–Garcia–Hauser[15] (EGH) have recently obtained new solutions by applying this transformation to some known colliding wave metrics, including the Chadrasekhar–Xanthopoulos,[9] Nutku–Halil,[16] and Ferrari–Ibanez–Bruni[17] solutions, which can all be generated from an appropriate Kasner seed metric using the BZ soliton technique. However, as pointed out by Letelier[3] and manifest in our case, this transformation is built into the BZ method and if one considers the general solutions one immediately covers the EGH generalizations.

(iii) Cylindrical waves. In the cylindrically symmetric gauge Eqs. (5.21) gives

$$\alpha = \rho = w[(1 - x^2)(1 - y^2)]^{1/2}, \quad \beta = t = wxy. \quad (5.27)$$

Again one is restricted to regions where either $|x| < 1$ and $|y| < 1$ or $|x| > 1$ and $|y| > 1$. In terms of the (t,ρ) coordinates, the inverse of Eq. (5.27), which reads as

$$2wx = r_+ + r_-, \quad 2wy = r_+ - r_-, \quad r_\pm \equiv [(t \pm w)^2 - \rho^2]^{1/2}, \quad (5.28)$$

shows that the solution given by Eqs. (5.11) and (5.23) is valid only in the three disconnected regions I, II, and III bounded by the symmetry axis $\rho = 0$ and the lines $t = |w| + \rho$, $t = \pm |w| \pm \rho$, and $t = -|w| - \rho$, respectively. In each of the regions I–III we have a metric representing cylindrical gravitational waves since the metric is time-dependent and cylindrically symmetric. However, one has to determine the fashion in which the metric extends beyond the $|x| = |y|$ lines before one has a clear picture of the physical interpretation of the line element given by Eqs. (5.11) and (5.23) in the cylindrically symmetric gauge. Work by the present authors regarding this point is under progress.

B. A pair of double poles

Starting with the same seed metric that was used in Sec. V A, a whole family of new solutions is obtained by simply assuming that the poles w_1 and w_2 are now double poles. This follows from the fact that in this case the results of Sec. IV B apply, whereby two more parameters enter the picture, namely ξ_1 and ξ_2. Just as the Q_k's of Eq. (5.3), these parameters must be chosen to be either real or complex conjugate when the poles (w_1,w_2) are real or complex conjugate, respectively, because otherwise the product metric coefficients will not be real.

Let it be noted that according to Eqs. (4.15), the elements of the pertinent $[4\times4]$ matrices $\{E_{(\delta)}\}$ have already been expressed in terms of the known functions (σ_1,σ_2) and the four parameters (Q_1,Q_2,ξ_1,ξ_2). Therefore, the calculation of the metric coefficients corresponding to the four-soliton case at hand is a matter of straightforward, if tedious, algebra. Since the resulting expressions for the intermediate L functions are very lengthy, we restrict ourselves to presenting only the product metric coefficients. As in the simple poles case presented in Sec. V A, the double-poles' solutions split into two branches corresponding to the w_k's being real or complex conjugate, respectively, as follows.

1. Complex conjugate poles

When $w_k\in\mathbb{C}$, the product metric is given by Eq. (5.11) where, now,

$$f=\text{const}[X/(x^2+y^2)^4]\,, \tag{5.29a}$$

$$Y=E^2+D^2\,, \tag{5.29b}$$

$$X=F^2+G^2\,, \tag{5.29c}$$

$$(\omega/4w)=(FH+GR)/Y+\text{const}\,, \tag{5.29d}$$

$$E\equiv p^2(x^2+1)^2-q^2(y^2-1)^2 - (r^2+s^2)(1+p^2)(x^2+y^2)^2\,, \tag{5.29e}$$

$$D\equiv 2[(x^2+1)(y^2-1)]^{1/2}\{-pq(x^2+y^2)+(y^2-x^2)\times(qr-lps)-2zy(qs+lpr)\}\,, \tag{5.29f}$$

$$F\equiv(x^2+1)(1-px)^2-(y^2-1)(l-qy)^2 - (1+p^2)(x^2+y^2)\{1+(r^2+s^2)(x^2+y^2) + 2(rx+sy)\}\,, \tag{5.29g}$$

$$G\equiv 2(x^2+1)\{[(p+r)x+sy](l-qy) + p(ry-sx)(q-ly)\} + 2(y^2-1)\{[(q+ls)y+lrx](1-px) + q(ry-sx)(p+x)\}\,, \tag{5.29h}$$

$$H\equiv q(p+x)(y^4-1)-p(q-ly)(x^4-1) - (x^2+y^2)\{[r(q-ly)+sp(qy-l)](x^2+1) + [qr(1-px)-ls(x+p)](y^2-1)\}\,, \tag{5.29i}$$

$$R\equiv py(x^4-1)-qlx(y^4-1) + 2yx[p^2(x^2+1)+q^2(y^2-1)] + (1+p^2)(x^2+y^2)[ry(x^2+1)+sx(y^2-1)]\,, \tag{5.29j}$$

and the real parameters r and s stand for the combinations

$$r=(w/2)(\xi_1+\xi_2)\,, \tag{5.30a}$$

$$s=(w/2i)(\xi_1-\xi_2)\,. \tag{5.30b}$$

Except for very particular choices for the values of the parameters involved, the metric coefficients given by Eqs. (5.29) share with their two-soliton analogs the same behavior on the (t,ρ) or (x,y) plane; therefore, the physical interpretation of the latter as described in Sec. V A *1* applies here as well. However, in the present case, the corresponding space-time structure is much richer than the one found in the two-soliton solutions. The solutions belonging to the axisymmetric gauge, for example, give the analog of the Kerr $a^2>m^2$ metric for the Kinnersley–Chitre[11] class of stationary axially symmetric space-times discussed below. However, a detailed analysis of the above solutions is required before an exact physical interpretaion is put forward: Since the same is true for the solutions that follow, it should be obvious that such an analysis cannot be presented in the context of the present paper.

2. Real poles

The metric coefficients of the four-soliton solution that results from the choice $w_k\in\mathbb{R}$ can be obtained from those corresponding to the case $w_k\in\mathbb{C}$ by the substitution

$$(x,w,p,r,s)\to(ix,-iw,-ip,-ir,-s)\,. \tag{5.31}$$

Equation (5.31) is a consequence of the pertinent formulas and a simple extension of the analogous result obtained in the two-soliton case. However, since no complete list of these coefficients has been published thus far, we prefer to give them here explicitly. Again, the line element has the form given by Eq. (5.11) where, now,

$$f=\text{const}[X/(x^2-y^2)^4]\,, \tag{5.32a}$$

$$Y=E^2-D^2\,, \tag{5.32b}$$

$$X=F^2+G^2\,, \tag{5.32c}$$

$$(\omega/4w)=-(FH+GR)/Y+\text{const}\,, \tag{5.32d}$$

$$E\equiv -p^2(x^2-1)^2-q^2(y^2-1)^2 + (r^2-s^2)(1-p^2)(x^2-y^2)^2\,, \tag{5.32e}$$

$$D\equiv 2[(x^2-1)(1-y^2)]^{1/2}\{pq(x^2-y^2)+(y^2+x^2)\times(qr+lps)-2xy(qs+lpr)\}\,, \tag{5.32f}$$

$$F\equiv -(x^2-1)(1-px)^2-(y^2-1)\times(l-qy)^2+(1-p^2)(x^2-y^2)\times\{1+(r^2-s^2)(x^2-y^2)+2(rx-sy)\}\,, \tag{5.32g}$$

$$G\equiv -2(x^2-1)\{[(p+r)x-sy]\times(l-qy)-p(ry-sx)(q-ly)\} + 2(y^2-1)\{[(q-ls)y+lrx](1-px) + q(ry-sx)(x-p)\}\,, \tag{5.32h}$$

$$H\equiv q(x-p)(y^4-1)+p(q-ly)(x^4-1) + (x^2-y^2)\{[r(q-ly)-sp(qy-l)](x^2-1) + [qr(1-px)-ls(x-p)](1-y^2)\}\,, \tag{5.32i}$$

$$R\equiv -py(x^4-1)-qlx(y^4-1) + 2yx[p^2(x^2-1)+q^2(y^2-1)] - (1-p^2)(x^2-y^2)[ry(x^2-1)-sx(y^2-1)]\,. \tag{5.32j}$$

Depending on the gauge, the following classes of solutions can be distinguished.

(i) In the axisymmetric gauge Eqs. (5.32) give the Kinnersley–Chitre[11] class of metrics: The latter represents a two-parameter generalization of the $\delta = 2$ Tomimatsu–Sato[12] class and was discovered using the symmetry transformations that leave the field equations invariant. Our method of deriving the same class of solutions verifies the Tomimatsu conjecture[8] that the Kinnersley–Chitre[11] metrics should be the product of letting the two poles that appear in the derivation of the $m^2 > a^2$ Kerr metric via the BZ technique to become double. Moreover, our method makes explicit a particular advantage of the BZ technique over the one used by Kinnersley and Chitre. This consists of the fact that the BZ method leads directly to all the components of the product metric, while the Kinnersley–Chitre method leads to the Ernst potential, which implies that some integrations must be performed before the metric is specified completely.

(ii) In the cylindrically symmetric gauge Eqs. (5.32) represent "cylindrical waves."

(iii) In the plane symmetric gauge Eqs. (5.32) represent "interacting plane waves." The necessity of the quotation marks derives, in cases (i) and (ii) from the fact that the corresponding solutions are valid in disconnected spacetime regions that are separated from each other by the null hypersurfaces $x^2 = y^2$ along which f is singular. Therefore, no claim to a concrete physical interpretation can be substantiated before any one of the above regions is appropriately extended. Given that Chandrasekhar and Xanthopoulos[9] have already shown that such an extension is not possible in the plane-waves version of the Tomimatsu–Sato[12] solution using the well-known Khan–Penrose technique, it seems that the five-parameters family presented above cannot fare better.

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